PHYSICAL REVIEW E VOLUME 51, NUMBER 5 MAY 1995

Universal spatiotemporal scaling in the dynamics of one-dimensional pattern selection

Michael I. Tribelsky,[1,*] Shoichi Kai,[2] Hideki Yamazaki,[3] and Manuel G. Velarde[4]
[1]*Institute for Mathematical Sciences, KAO Corporation, 2-1-3, Bunka, Sumida-ku, Tokyo, 131, Japan*
[2]*Department of Applied Physics, Kyushu University, Fukuoka, 812, Japan*
[3]*Department of Electronics, Fukuoka Institute of Technology, Fukuoka, Waziro, 811-02, Japan*
[4]*Instituto Pluridisciplinar Universidad Complutense de Madrid, Paseo Juan XXIII, no. 1, 28040 Madrid, Spain*
(Received 6 July 1994; revised manuscript received 26 January 1995)

It is shown that the dynamics of pattern selection in quasi-one-dimensional extended systems may be described as a discrete process of alteration of the number of points where an order parameter of the system vanishes. Close to the alteration moment, the system has a universal spatiotemporal behavior. The one-dimensional Swift-Hohenberg and Ginzburg-Landau equations are considered as examples. Both yield a spatiotemporal scaling with the same universal exponent.

PACS number(s): 47.20.Ky, 47.65.+a, 61.30.Gd

The dynamics of pattern selection is an important problem in a wide variety of different extended systems, such as convection of both isotropic and anisotropic fluids, Taylor-Couette flow, directional solidification, and many others [1]. The goal of the present paper is to show that in the quasi-one-dimensional case, close to certain singled-out points in the (x-t) plane, the dynamics becomes independent of the details of the initial conditions, while characteristic spatial and temporal scales of the problem begin to depend on each other, being connected by a scaling law.

Let us consider, for example, the one-dimensional (1D) Swift-Hohenberg (SH) equation [2]

$$\frac{\partial u}{\partial t} = \left[\epsilon - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right] u - u^3. \qquad (1)$$

At real u and small real ϵ the SH equation is a universal equation describing the dynamics of a number of extended dissipative systems close to a threshold of pattern formation [3]. The meaning of u and ϵ depends on the formulation of each concrete problem. For instance, in the case of Rayleigh-Bénard convection in a horizontal layer for which the equation was initially introduced in Ref. [2], ϵ is a rescaled deviation of the Rayleigh number from the threshold of the convective instability (reduced control parameter) and u stands for the vertical fluid velocity at the midlane, so that the points $x_n(t)$ where u vanishes (nodes) correspond to the centers of convection rolls.

At $\epsilon = \text{const}$, $0 < \epsilon \ll 1$, Eq. (1) has a family of steady spatially periodic solutions whose principle wave number k ($k \equiv 2\pi/\lambda$, where λ stands for the spatial period) satisfies the condition $|1 - k| \leq \sqrt{\epsilon}/2$. The solutions with k lying outside the segment $|1 - k| \leq \sqrt{\epsilon/12}$ are unstable with respect to infinitesimal perturbations: If such an unstable solution is taken as the initial condition, Eckhaus instability (EI) occurs [4]. The EI development finally transforms the initial unstable solution into a stable one that belongs to the same family, but has another value of k [5].

It is evident that any wave number alteration process can be the case only if the dynamics results in a change of the total number of the nodes. The latter takes place during a passage of a local extremum of u via the x axis; see Fig. 1(a). Such a scenario corresponds to the well-known process of creation or annihilation of a pair of convection rolls (0↔2) without reconnection of stream lines [6]. However, in degenerate cases positions of some nodes may be fixed due to the symmetry. If such a fixed node is engaged in the process, the dynamics is different.

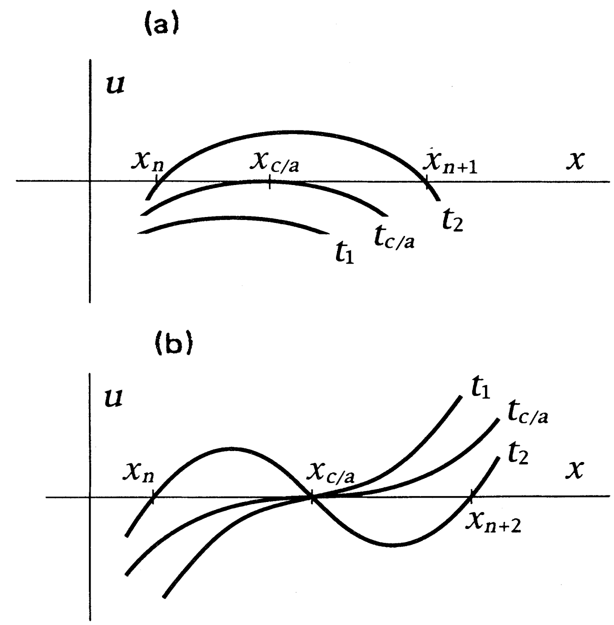

FIG. 1. Dynamics of creation (annihilation) of a pair of rolls within the framework of the SH equation (qualitatively). $t_{1,c/a,2}$ stand for different moments of time: $t_1 < t_{c/a} < t_2$, creation; $t_1 > t_{c/a} > t_2$, annihilation. (a) Passage of a local extremum via x axis. (b) Degenerate case. Two nodes at $x = x_{n,n+2}$ detach from (merge with) one at $x = x_{c/a}$, whose position is fixed by the symmetry.

*Electronic address: tribel@kisi.kao.co.jp

1063-651X/95/51(5)/5132(4)/$06.00 51 5132

In this case one roll splits into three (creation) or three rolls merge into one (annihilation), with a reconnection of stream lines via a stagnation point (1↔3); see Fig. 1(b) [7].

Let us study the dynamics of the order parameter evolution close to a creation or annihilation point $(x_{c/a}, t_{c/a})$. Being a regular function of x and t, the order parameter may be expended in powers of $\Delta x \equiv x - x_{c/a}$, $\Delta t \equiv t - t_{c/a}$. The expansion reads as follows:

$$u = u_{01}\Delta t + \frac{1}{2}u_{20}(\Delta x)^2 + \text{h.o.t. for } 0 \leftrightarrow 2,$$
$$u = u_{11}\Delta t \Delta x + \frac{1}{3!}u_{30}(\Delta x)^3 + \text{h.o.t. for } 1 \leftrightarrow 3, \tag{2}$$

where h.o.t. denotes higher-order terms. Equations (2) both yield the same scaling:

$$\Delta x_{\rm PS} = \sqrt{\alpha \Delta t} + O(\Delta t), \tag{3}$$

at small $|\Delta x_{\rm PS}|$ and $|\Delta t|$. Here $\Delta x_{\rm PS}$ is the distance between splitting (merging) nodes [the meaning of the index PS will be clear later; cf. below Eq. (6)]. The prefactor α is determined from the interplay of coefficients in Eqs. (2) and may be both positive (creation) and negative (annihilation).

Relation (3) is the desirable scaling law. Note that the critical exponent in Eq. (3) is 1/2 in spite of the *fourth*-order differential operator in the SH equation.

On the other hand, at small ϵ Eq. (1) may be reduced to the Ginzburg-Landau (GL) equation

$$\psi_T = \psi_{XX} + (1 - |\psi|^2)\psi, \tag{4}$$

where $u = (\psi e^{ix} + c.c.)\sqrt{\epsilon/3}$ and X, and T stand for "slow" variables $X \equiv x\epsilon^{1/2}/2$, and $T \equiv \epsilon t$ [3]. Within the framework of the GL equation the creation or annihilation process occurs when $|\psi|$ vanishes at a certain point $X = X_{\rm PS}$, $T = T_{\rm PS}$ and its phase $\varphi \equiv \arg\psi$ has a discontinuity (see, e.g., Refs. [8,9]). This phenomenon, known as the phase slip (PS) process, also plays a very important role in the so-called resistive state of superconductors [10]. In what follows we discuss the dynamics of the PS process from a viewpoint that makes its nature extremely clear.

Let us consider the U-V plane, where $U \equiv \text{Re}\psi$ and $V \equiv \text{Im}\psi$, and the same phase trajectory of the system as a function of X at two different fixed moments of time — just before the PS and just after it; see Fig. 2. Departures of X and T from the PS point $\Delta X \equiv X - X_{\rm PS}$ and $\Delta T \equiv T - T_{\rm PS}$ are supposed to be small. By definition, at the PS point $U(X_{\rm PS}, T_{\rm PS}) = V(X_{\rm PS}, T_{\rm PS}) = 0$, i.e., at $\Delta T = 0$, the phase trajectory passes via the origin of the coordinate frame in the U-V plane. Note, however, that in spite of the similarity in shape, the curves depicted in Fig. 2 have a drastic difference in their properties: At any fixed *negative* ΔT phase, φ is a *monotonic* function of ΔX. A variation of ΔX from a small negative value to a small positive one gives rise to an *increase* in φ of a quantity close to π ($\Delta\varphi_1 \approx \pi$). On the other hand, at any fixed *positive* ΔT the trajectory has two tangents passing through the origin of the coordinate frame. It results in a *nonmonotonic* behavior of φ as a function of ΔX: It reaches extrema each time the radius vector coincides with the tangents. Besides, now a variation of ΔX from $\Delta X_1 < 0$ (the first point of tangency — lower half plane in Fig. 2) to $\Delta X_2 > 0$ (the second one — upper half plane) *decreases* φ by nearly π ($\Delta\varphi_2 \approx -\pi$). The closer the trajectory to the origin of the coordinate frame, the closer $|\Delta\varphi_{1,2}|$ to π and $\Delta X_{1,2}$ to zero. Thus, the passage of the trajectory via the origin of the coordinate frame in the U-V plane generates the phase shift of -2π [11].

Supposing U and V to be regular functions of X and T at the PS point, expanding them in powers of $\Delta X, \Delta T$, and taking into account that the expansion must satisfy Eq. (4), it is easy to find that at fixed ΔT the extrema of φ are located at $\Delta X_{1,2} = \pm\sqrt{2\Delta T} + O(\Delta T)$ and the phase difference caused by a variation of ΔX from ΔX_1 to ΔX_2 is

$$|\Delta\varphi_2| = \pi + O(\sqrt{\Delta T}). \tag{5}$$

It is worth mentioning that $|\Delta\varphi_2|$ is always close to, but *smaller* than, π (see Fig. 2); thus $O(\sqrt{\Delta T})$ designates a *negative* correction to Eq. (5). It is natural to define the distance between two extrema of φ, i.e.,

$$\Delta X_{\rm PS} = 2\sqrt{2\Delta T} + O(\Delta T)\ , \tag{6}$$

as the width of the PS core — a region of sharp phase variation.

Earlier we (M.I.T., S.K., and H.Y.) obtained for the PS process in the GL equation $\Delta X_{\rm PS} \sim \sqrt{|\Delta T|}$ [8]. Now one can see that the prefactor $2\sqrt{2}$ also is a universal constants of the problem. The universality of the prefactor in Eq. (6) makes the PS process in the GL equation *strictly anisotropic* in time: At fixed ΔT close to a PS point, the dependence $\varphi(\Delta X)$ is always monotonic at $\Delta T < 0$ and has two extrema at $\Delta T > 0$ that coincide with results of a numerical simulation of the process [8,9].

In other words, the PS process in the GL equation is strictly connected with the sign of curvature of the phase trajectory close to the origin of the U-V plane: It

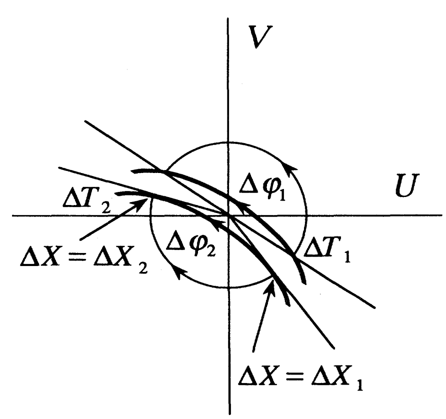

FIG. 2. Same phase trajectory (bold lines) in the U-V plane close to the PS point at two different fixed moments of time (schematically); $\Delta T_1 < 0$ and $\Delta T_2 > 0$. Arrows on the trajectory designate the direction of increase of ΔX.

is in the future if the center of the curvature associated with the normal, passing via the origin of the U-V plane, and the origin itself both are located on the same side with respect to the phase trajectory, and in past if the trajectory passes between them; see Fig. 2.

Note that the SH and the GL equations both have the same scaling index 1/2 [cf. Eqs. (3) and (6)] despite different orders of these equations. Taking into account that X, T are *slow* variables, we can conclude that the same act of creation or annihilation is characterized by *two* scaling laws related to Eqs. (3) and (6), respectively. Namely, Eq. (3) describes a *fast* relaxation of perturbations induced by the creation or annihilation of a pair of rolls. Only a few rolls are engaged into the process and the scaling is valid until $\Delta x_{\rm PS}$ is small compared to the characteristic diameter of rolls far from the creation or annihilation point (local, or *microscopic*, scale). Meanwhile Eq. (6) is associated with a *slow* relaxation of the GL phase. The characteristic spatial scale now is of order $\epsilon^{-1/2}$ and a large number of rolls are involved into the process (global, or *macroscopic*, scale). To get this global scaling one must obtain the GL phase φ treating the solution of the SH equation. There are several possibilities for the treatment, say, employing a Fourier transform [9,12]. The procedure is well known and will not be discussed here.

A numerical simulation of EI in the SH equation agrees with Eqs. (3) and (6) including the value of the prefactor in Eq. (6) [13]. The same result is found for the simulation of the PS dynamics in the GL equation reported in Refs. [8,9].

To obtain experimental evidence of the scalings, electrohydrodynamic convection (EHC) in the nematic liquid crystal N-(4′-methoxybenzylidene)-4-(n-butyl)aniline was employed. EI in EHC was realized by the frequency-voltage jump method whose details were already described in Refs. [9,14]. In our case the critical and the operating frequencies were 1200 Hz and 100 Hz, respectively. The sample used had the aspect ratios (the lateral dimensions to the thickness of fluid layer) $\Gamma_x = 200$ and $\Gamma_y = 2$, which made the system 1D. The unstable 1D patterns, created as the initial conditions in the experiments, consisted of about 100 pairs of rolls aligned in the x direction so that their axes were perpendicular to the long sidewall of the sample. The unstable patterns always had a spatial period *smaller* than that for the stable patterns. Thus EI development could result in *annihilation* processes only. The temporal evolution of an optical image of the patterns was analyzed by an image grabber (Nexus Qube) electronic system and a personal computer (NEC Model PC-9801VM). Typical results of this analysis are shown in Fig. 3, where $\epsilon = (V^2 - V_c^2)/V_c^2$. Both annihilation processes—normal ($2 \to 0$) and degenerate ($3 \to 1$)—were observed clearly. It is worth mentioning that the degenerate case occurred at rather large values of ϵ ($\epsilon = 0.3$ and 0.5), while the normal process was commonly observed at $\epsilon = 0.1$. The distance $\Delta x_{\rm PS}$ between the centers of annihilating rolls (the *microscopic* scale) was obtained from the images. The dynamics for *macroscopic* aspects $[\Delta X(T)_{\rm PS}]$ was studied by analyzing the spectrum of the fast Fourier transform in the same manner as that in Refs. [9,12]. At negative ΔT (just before the PS) the quantity $\Delta X_{\rm PS}$ was determined as the length of the smallest segment with the phase difference of π between its edges. The results are shown in Fig. 4. The experimental data fit straight lines at $(\Delta x_{\rm PS}/\lambda_0)^2 < 0.2$ [Fig. 4(a)] and $(\Delta X_{\rm PS}/\lambda_0)^2 < 6$

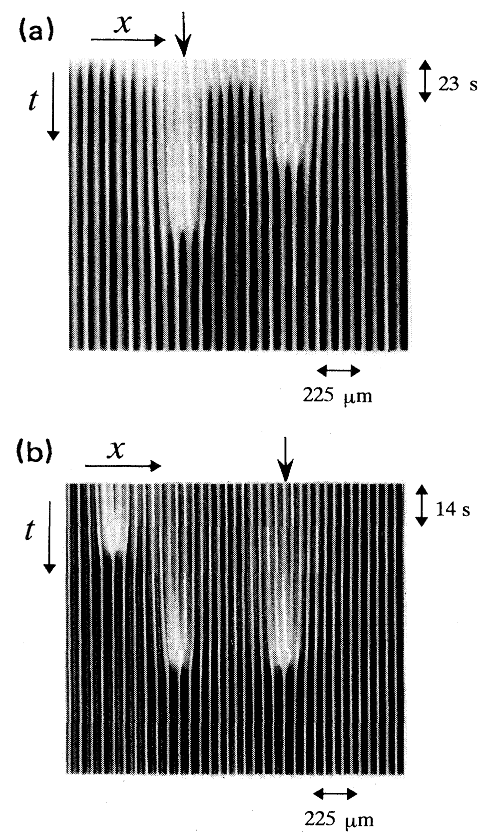

FIG. 3. Spatiotemporal change of an image of 1D patterns during EI development (EHC in nematics — experiment). The interval between each of the two white lines corresponds to the local value of the roll diameter. The origins in time and space axes are arbitrary. Arrows indicate $x_{c/a}$ for the most clearly expressed annihilation processes of each type: (a) transition $2 \to 0$, $\epsilon = 0.1$; (b) transition $3 \to 1$, $\epsilon = 0.3$.

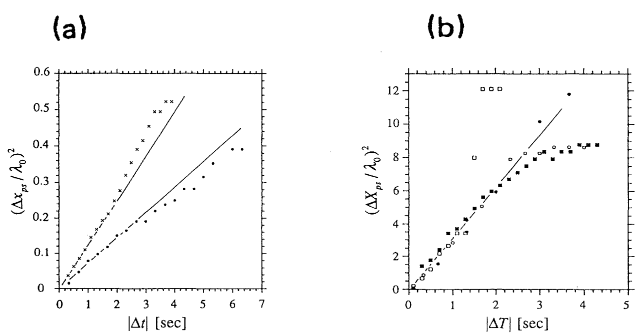

FIG. 4. Treatment of the results shown in Fig. 3. λ_0 (initial wavelength) is 53 μm at $\epsilon = 0.1$ and 43 μm at $\epsilon = 0.3$. (a) *Microscopic* scaling relation ($\Delta x_{\rm PS}$-Δt). •, ϵ=0.1; ×, 0.3. (b) *Macroscopic* scaling relation ($\Delta X_{\rm PS} - \Delta T$). ○, $\epsilon = 0.1$ and $\Delta T < 0$; •, $\epsilon = 0.1$ and $\Delta T > 0$; □, $\epsilon = 0.3$ and $\Delta T < 0$; ■, $\epsilon = 0.3$, and $\Delta T > 0$.

[Fig. 4(b)]. The slope of the lines in Fig. 4(a) depends on the value of ϵ. Besides, even at the same ϵ it is different at different realizations of the annihilation process, which agrees with the above mentioned nonuniversality of the prefactor α in Eq. (3). In Fig. 4(b) the slope is universal within the accuracy of our experiment [cf. Eq. (6)]. Thus both *microscopic* [Eq. (3)] and *macroscopic* [Eq. (6)] scalings were well proved experimentally.

Summarizing the results of our analysis we would like to emphasize the following.

(1) Scaling $\Delta x \sim| \Delta t |^{1/2}$ is a generic property of a variety of different problems that *does not depend on the order of differential operator* in the corresponding governing equation [cf. Eq. (3) to the SH equation and Eq. (6) to the GL equation].

(ii) The only ground for this universality is connected with an opportunity to find (a) a *discrete* quantity associated with a *continuous* order parameter [such as the total number of nodes for $u(x,t)$] and (b) a relevant representation of the order parameter, so that it remains a regular function of x and t at the bifurcation moment when the discrete quantity alters.

(iii) The value of the critical exponent (1/2) has no relation to phase diffusion, or to any other diffusionlike processes, being completely determined by *local* characteristics of the order parameter close to the bifurcation point.

(iv) If the system has several different spatiotemporal scales (such as the SH equation), each one can bring about its own scaling.

One of us (M.I.T.) is grateful to Ministerio de Educación y Ciencia, Spain for financial support for his stay at Instituto Pluridisciplinar, Universidad Complutense de Madrid where this work was begun, to Juan Millán-Rodríguez and Carlos Pérez-García for valuable discussions during his (M.I.T.) visit to Universidad de Navarra, and to the entire staff of the Department of Physics and Applied Mathematics, Universidad de Navarra for extremely warm hospitality. This work is partially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture, Japan.

[1] For recent results see, e.g., *Pattern Formation in Complex Dissipative Systems*, edited by S. Kai (World Scientific, Singapore, 1992).

[2] J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).

[3] See, e.g., M. I. Tribelsky, Izv. Akad. SSSR, Nauk Ser. Fiz. **53**, 722 (1989) [Bull. Acad. Sci. USSR, Phys. Ser. **53**, 107 (1989)].

[4] W. Eckhaus, *Studies in Nonlinear Stability Theory* (Springer-Verlag, Berlin, 1965).

[5] A similar transition between two locally stable states may be induced by a finite-amplitude perturbation; see M. I. Tribelsky, S. Kai, and H. Yamazaki, Prog. Theor. Phys. **86**, 963 (1991). All results concerning the scaling in the case of the Eckhaus instability development may be applied to this transition also.

[6] M. Lowe and J. P. Gollub, Phys. Rev. Lett. **55**, 2575 (1985).

[7] Note that while transition $1 \leftrightarrow 3$ is an interesting complement to process $0 \leftrightarrow 2$, as far as we know, it has never been discussed before.

[8] M. I. Tribelsky, S. Kai, and H. Yamazaki, Phys. Rev. A **45**, 4175 (1992).

[9] H. Yamazaki, M. I. Tribelsky, S. Nasuno, and S. Kai, in *Pattern Formation in Complex Dissipative Systems* (Ref. [1]), pp. 346–357.

[10] See, e.g., L. Kramer and A. Baratoff, Phys. Rev. Lett. **38**, 518 (1977); B. I. Ivlev and N. B. Kopnin, Adv. Phys. **33**, 47 (1984); S. M. Golberg, N. B. Kopnin, and M. I. Tribelsky, Zh. Eksp. Teor. Fiz. **94**, 289 (1988) [Sov. Phys. JETP **67**, 812 (1988)]; J. Low Temp. Phys. **77**, 209 (1989), and references therein.

[11] The fact that the phase shift of 2π is related to a simple zero of the phase trajectory was mentioned earlier by Lorenz Kramer (private communication).

[12] G. Goren, I. Procaccia, S. Rasenat, and V. Steinberg, Phys. Rev. Lett. **63**, 1237 (1989).

[13] J. Millán-Rodrìguez and C. Pérez-Garcìa (private communication).

[14] S. Nasuno, O. Sasaki, S. Kai, and W. Zimmermann, Phys. Rev. A **46**, 4954 (1992).

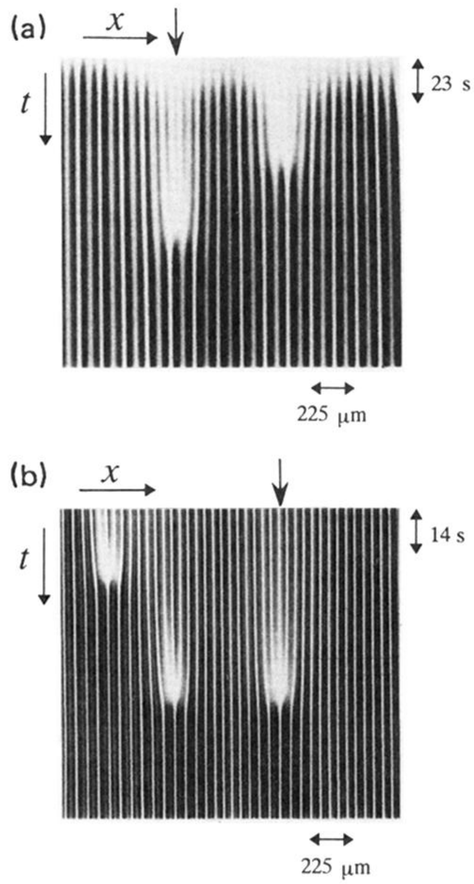

FIG. 3. Spatiotemporal change of an image of 1D patterns during EI development (EHC in nematics — experiment). The interval between each of the two white lines corresponds to the local value of the roll diameter. The origins in time and space axes are arbitrary. Arrows indicate $x_{c/a}$ for the most clearly expressed annihilation processes of each type: (a) transition $2 \rightarrow 0$, $\epsilon = 0.1$; (b) transition $3 \rightarrow 1$, $\epsilon = 0.3$.